Averaging principle for slow-fast stochastic system driven by α-stable processes

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The 16th Workshop on Markov Processes and Related Topics, BNU and CSU

Outline

- Backgroud
- Main results
- Idea of proof

1. Backgroud

Many slow-fast (also called multiscale or two-time scales) system arise from material sciences, chemistry, fluids dynamics, biology and other application areas, such as

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1. Backgroud

Many slow-fast (also called multiscale or two-time scales) system arise from material sciences, chemistry, fluids dynamics, biology and other application areas, such as

- In climate models, where climate-weather interactions may be studied within the averaging framework, climate being the slow motion and weather the fast one.
- In the chemistry, the dynamics of chemical reaction networks often take place on notably different times scales, from the order of nanoseconds (10^{-9} s) to the order of several days.

Averaging principle for SDEs (By Khasminskii, 1968)

$$\begin{cases} dX_t^\varepsilon = b(X_t^\varepsilon, Y_t^\varepsilon)dt + \sigma(X_t^\varepsilon, Y_t^\varepsilon)dW_t, \quad X_0^\varepsilon = x \in \mathbb{R}^d, \\ dY_t^\varepsilon = \frac{1}{\varepsilon} f(X_t^\varepsilon, Y_t^\varepsilon)dt + \frac{1}{\sqrt{\varepsilon}} g(X_t^\varepsilon, Y_t^\varepsilon)dW_t, \quad Y_0^\varepsilon = y \in \mathbb{R}^d. \end{cases}$$

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Assume that $\exists \bar{b}(x) : \mathbb{R}^d \to \mathbb{R}^d$, $A(x) : \mathbb{R}^d \to \mathbb{R}^{d\times d}$:

$$\left| \frac{1}{T} \int_0^T \mathbb{E} b(x, Y_t^{x,y}) dt - \bar{b}(x) \right| \to 0, \quad \varepsilon \to 0;$$

$$\left| \frac{1}{T} \int_0^T \mathbb{E} \sigma(x, Y_t^{x,y}) \sigma^*(x, Y_t^{x,y}) dt - A(x) \right| \to 0, \quad \varepsilon \to 0;$$

where $\{Y_t^{x,y}\}_{t\geq 0}$ is the unique solution of the frozen equation:

$$dY_t^{x,y} = f(x, Y_t^{x,y})dt + g(x, Y_t^{x,y})dW_t, \quad Y_0^{x,y} = y.$$

Averaging principle says:

$$X^{\varepsilon} \to \bar{X}, \quad \text{in weak sense,}$$

as $\varepsilon \to 0$, where $\bar{X}$ is the solution of the averaged equation:

$$d\bar{X}_t = \bar{b}(\bar{X}_t)dt + \bar{\sigma}(\bar{X}_t)dW_t, \quad X_0 = x.$$

where $\bar{\sigma}(x) := \sqrt{A(x)}$.

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where $\bar{\sigma}(x) := \sqrt{A(x)}$.
If the frozen equation admits a unique invariant measureμ^{x}. Then

- $\bar{b}(x) = \int_{\mathbb{R}^d} b(x, y)\mu^{x}(dy)$
- $\bar{\sigma}(x)\bar{\sigma}(x)^* = \int \sigma(x, y)\sigma(x, y)^*\mu^{x}(dy)$

An simple case: If $f(x,y) \equiv f(y)$ and $g(x,y) \equiv g(y)$,

$$
\begin{aligned}
Y^{\varepsilon}_{t\varepsilon} &= y + \frac{1}{\varepsilon}\int_0^{t\varepsilon} f(Y^{\varepsilon}_s)ds + \frac{1}{\sqrt{\varepsilon}}\int_0^{t\varepsilon} g(Y^{\varepsilon}_s)dW_s \\
&= y + \int_0^t f(Y^{\varepsilon}_{r\varepsilon})dr + \int_0^t g(Y^{\varepsilon}_{r\varepsilon})d\tilde{W}_r,
\end{aligned}
$$

where $\tilde{W}_r := \frac{1}{\sqrt{\varepsilon}} W_{r\varepsilon}$ is also a Brownian motion.

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where $\tilde{W}_r := \frac{1}{\sqrt{\varepsilon}}W_{r\varepsilon}$ is also a Brownian motion.
Based on the uniqueness of solutions of the frozen equation:

$$Y_t = y+\int_0^t f(Y_r)dr+\int_0^t g(Y_r)dW_r$$

$$\Rightarrow \quad \mathbb{P}\circ(Y^{\varepsilon}_{t\varepsilon})^{-1}=\mathbb{P}\circ(Y_t)^{-1}$$

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$$\Rightarrow \quad \lim_{\varepsilon\to 0}\mathbb{P}\circ(Y^{\varepsilon}_{t})^{-1}=\lim_{s\to\infty}\mathbb{P}\circ(Y_s)^{-1}=\mu \quad \text{(invariant measure).}$$

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Convergence rate? Whether optimal? In the Wiener noise case

- Strong sense: $\left[\sup_{t\in[0,T]} \mathbb{E}|X_t^\varepsilon - \bar{X}_t|^p\right]^{1/p} \leq C_T \varepsilon^{1/2}$
- Weak sense: $\sup_{t\in[0,T]} |\mathbb{E}\varphi(X_t^\varepsilon) - \mathbb{E}\varphi(\bar{X}_t)| \leq C_T \varepsilon$

The main tools:

- Khasminskii's time discretization
- Asymptotic expansion of the solutions of Kolmogorov equation with respect to ε
- Poisson equation

The Khasminskii's time discretization

- A.Y. Veretennikov, On the averaging principle for systems of stochastic differential equations, Math. USSR Sborn. 1991.
- S. Cerrai, A Khasminskii type averaging principle for stochastic reaction-diffusion equations, AAP, 2009
- D. Liu, Strong convergence of principle of averaging for multiscale dynamical systems, Commun. Math. Sci., 2010
- H. Fu, J. Liu, Strong convergence in stochastic averaging principle for two time-scales stochastic partial differential equations, JMAA, 2011
- W. Wang, A.J. Roberts, Average and deviation for slow-fast stochastic partial differential equations, JDE, 2012

- B. Pei, Y. Xu, G. Yin, Stochastic averaging for a class of two-time-scale systems of stochastic partial differential equations, Nonlinear Anal., 2017
- P. Gao, Averaging principle for the higher order nonlinear Schrödinger equation with a random fast oscillation, JSP, 2018
- W. Liu, M. Röckner, X. Sun, Y. Xie, Averaging principle for slow-fast stochastic differential equations with time dependent locally Lipschitz coefficients, JDE, 2020
- X. Sun, L. Xie, Y. Xie, Averaging principle for slow-fast stochastic partial differential equations with Hölder continuous coefficients. JDE, 2021

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Asymptotic expansion of the solutions of Kolmogorov equation with respect to ε

- R. Z. Khasminskii, G. Yin, On averaging principles: an asymptotic expansion approach, SIAM JMA, 2004
- C.E.Brehier, Strong and weak orders in averaging for SPDEs, SPA, 2012
- H. Fu, L. Wan, J. Liu, X. Liu, Weak order in averaging principle for stochastic wave equation with a fast oscillation, SPA, 2018
- Z. Dong, X. Sun, H. Xiao, J. Zhai, Averaging principle for one dimensional stochastic Burgers equation. JDE, 2018

  $\cdots\ \cdots$

The Poisson equation

- E. Pardoux, A. Yu. Veretennikov, On the Poisson equation and diffusion approximation. AOP, 2001,2003
- S. Cerrai, M. Freidlin, Averaging principle for a class of stochastic reaction-diffusion equations. PTRF, 2009
- C.E. Bréhier, Orders of convergence in the averaging principle for SPDEs: the case of a stochastically forced slow component, SPA, 2020
- M. Röckner, X. Sun, Y. Xie, Strong convergence order for slow-fast McKean-Vlasov stochastic differential equations, AIHP, 2021
- M. Röckner, L. Xie, Diffusion approximation for fully coupled stochastic differential equations, AOP, 2021

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The papers mentioned above mostly considered the Wiener noise. How about the case of jump noise?

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- D. Givon, Strong convergence rate for two-time-scale jump-diffusion stochastic differential systems, SIAM J. Multiscale Model. Simul., 2007
- D. Liu, Strong convergence rate of principle of averaging for jump-diffusion processes, Front. Math. China, 2012
- J. Xu, L^p-strong convergence of the averaging principle for slow-fast SPDEs with jumps, JMAA, 2017
- B. Pei, Y. Xu, J. L. Wu, Two-time-scales hyperbolic-parabolic equations driven by Poisson random measures: Existence, uniqueness and averaging principles, JMAA, 2017

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However, the above jump noise excludes the α-stable noise, which has the heavy tail property and has many application in physics, finance and other fields.

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- J. Bao, G. Yin, C. Yuan, Two-time-scale stochastic partial differential equations driven by α-stable noises: Averaging principles, Bernoulli, 2017
- X. Sun, J. Zhai, Averaging principle for stochastic real Ginzburg-Landau equation driven by α-stable process, CPAA, 2020
- Y. Chen, Y. Shi, X. Sun, Averaging principle for slow-fast stochastic Burgers equation driven by α-stable process. Appl. Math. Lett. 2020

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But, no satisfactory convergence rates were obtained. Question:

- What are the optimal strong and weak convergence rates?
- Will it depends on the index α? How it depends?

2. Main results

- SDE case:

$$\begin{cases} dX_t^\varepsilon = b(X_t^\varepsilon, Y_t^\varepsilon)dt + dL_t^1, \quad X_0^\varepsilon = x \in \mathbb{R}^{d_1}, \\ dY_t^\varepsilon = \frac{1}{\varepsilon} f(X_t^\varepsilon, Y_t^\varepsilon)dt + \frac{1}{\varepsilon^{1/\alpha}} dL_t^2, \quad Y_0^\varepsilon = y \in \mathbb{R}^{d_2}, \end{cases} \tag{1}$$

where $\{L_t^1\}_{t\geq 0}$ and $\{L_t^2\}$ are independent d_1 and d_2 dimensional isotropic α-stable processes with $\alpha \in (1,2)$. $b : \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \to \mathbb{R}^{d_1}$ and $f : \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \to \mathbb{R}^{d_2}$.

Theorem 1(S., L. Xie, Y. Xie, Bernoulli, 2021)

(i) For any $(x, y) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, $T > 0$ and $p \in [1, \alpha)$, we have

$$\left(\mathbb{E} \sup_{t\in[0,T]} |X_t^\varepsilon - \bar{X}_t|^p \right)^{1/p} \leq C\varepsilon^{(1-1/\alpha)}. \quad (2)$$

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(ii) For any $\phi \in C_b^{2+\gamma}$ with $\gamma \in (\alpha - 1, 1)$,

$$\sup_{t\in[0,T]} |\mathbb{E}\phi(X_t^\varepsilon) - \mathbb{E}\phi(\bar{X}_t)| \leq C\varepsilon, \tag{3}$$

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where $\bar{X}$ is the solution of the averaged equation:

$$d\bar{X}_t = \bar{b}(\bar{X}_t)dt + dL_t^1, \quad \bar{X}_0 = x, \tag{4}$$

where $\bar{b}(x) = \int_{\mathbb{R}^{d_2}} b(x, y)\mu^x(dy)$.

- SPDE case:

$$\begin{cases} dX_t^\varepsilon = [AX_t^\varepsilon + B(X_t^\varepsilon, Y_t^\varepsilon)]\,dt + dL_t, \quad X_0^\varepsilon = x \in H, \\ dY_t^\varepsilon = \frac{1}{\varepsilon}[AY_t^\varepsilon + F(X_t^\varepsilon, Y_t^\varepsilon)]dt + \frac{1}{\varepsilon^{1/\alpha}} dZ_t, \quad Y_0^\varepsilon = y \in H, \end{cases} \tag{5}$$

where A is a selfadjoint operator, $B, F : H \times H \to H$ and $\{L_t\}_{t\geq 0}$ and $\{Z_t\}_{t\geq 0}$ be mutually independent cylindrical α-stable processes, where $\alpha \in (1,2)$, i.e.,

$$L_t = \sum_{k\in\mathbb{N}_+} \beta_k L_t^k e_k, \quad Z_t = \sum_{k\in\mathbb{N}_+} \gamma_k Z_t^k e_k, \quad t \geq 0.$$

Theorem 2(S., Y. Xie, arXiv:2106.02854, 2021)
(i) For any for any $(x, y) \in H^\eta \times H$ with $\eta \in (0,1)$, $T > 0$, $1 \le p < \alpha$ and small enough $\varepsilon, \delta > 0$,

$$\left(\sup_{t\in[0,T]} \mathbb{E}|X_t^\varepsilon - \bar{X}_t|^p\right)^{1/p} \le C_{T,\delta}\left[1 + \|x\|_\eta^{(1+\delta)} + |y|^{(1+\delta)}\right]\varepsilon^{1-\frac{1}{\alpha}}. \quad (6)$$

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(ii) For any test function $\phi \in C_b^3(H)$,$(x, y) \in H \times H$, $r \in (0, 1)$,

$$\sup_{t\in[0,T]} \left| \mathbb{E}\phi(X_t^\varepsilon) - \mathbb{E}\phi(\bar{X}_t) \right| \leq C_{r,T,\delta} \left[1 + |x|^{1+\delta} + |y|^{1+\delta} \right] \varepsilon^{1-r}, \quad (7)$$

where $\bar{X}$ is the solution of the averaged equation.

Example: Consider

$$\begin{cases} dX_t^\varepsilon = Y_t^\varepsilon dt + dL_t^1, \quad X_0^\varepsilon = x \in \mathbb{R}, \\ dY_t^\varepsilon = -\frac{1}{\varepsilon} Y_t^\varepsilon dt + \frac{1}{\varepsilon^{1/\alpha}} dL_t^2, \quad Y_0^\varepsilon = 0 \in \mathbb{R}, \end{cases}$$

where $\{L_t^1\}_{t\geq 0}$ and $\{L_t^2\}_{t\geq 0}$ are independent 1-dimensional symmetric α-stable process.

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where $\{L_t^1\}_{t\geq 0}$ and $\{L_t^2\}_{t\geq 0}$ are independent 1-dimensional symmetric α-stable process.

Thus the solution is given by

$$\begin{cases} X_t^\varepsilon = x + \int_0^t Y_s^\varepsilon ds + L_t^1, \\ Y_t^\varepsilon = \frac{1}{\varepsilon^{1/\alpha}} \int_0^t e^{-(t-s)/\varepsilon} dL_s^2. \end{cases}$$

Note that the corresponding frozen equation is

$$dY_t = -Y_t dt + dL_t^2, \quad Y_0 = 0$$

has a unique solution $Y_t = \int_0^t e^{-(t-s)} dL_s^2$, which admits a unique invariant measure μ with zero mean.

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has a unique solution $Y_t = \int_0^t e^{-(t-s)} dL_s^2$, which admits a unique invariant measure μ with zero mean.
Thus, the corresponding averaged equation is given by

$$\bar{X}_t = x + L_t^1.$$

As a result, we have for $0 < p < \alpha$,

$$\mathbb{E}|X_t^\varepsilon - \bar{X}_t|^p = \mathbb{E}\left|\int_0^t Y_s^\varepsilon ds\right|^p.$$

$$Z_t^\varepsilon := \int_0^t Y_s^\varepsilon ds = \frac{1}{\varepsilon^{1/\alpha}} \int_0^t \left[\int_r^t e^{-\frac{1}{\varepsilon}(s-r)} ds \right] dL_r^2.$$

As a result, the characteristic function of Z_t^ε is given by

$$\mathbb{E}\left(e^{ihZ_t^\varepsilon}\right) = \exp\left\{ -\int_0^t C_\alpha (1 - e^{-\frac{r}{\varepsilon}})^\alpha dr \left(\varepsilon^{1-1/\alpha}\right)^\alpha |h|^\alpha \right\}, \quad h \in \mathbb{R}.$$

where $\psi(x) = -C_\alpha |x|^\alpha$.

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where $\psi(x) = -C_\alpha|x|^\alpha$. Thus,

$$\mathbb{E}\left|\int_0^t Y_s^\varepsilon ds\right|^p = C_{\alpha,p}\left[\int_0^t (1-e^{-\frac{r}{\varepsilon}})^\alpha dr\right]^{p/\alpha} \left(\varepsilon^{1-\frac{1}{\alpha}}\right)^p,$$

which implies $1-\frac{1}{\alpha}$ is the optimal strong convergence rate.

3. Idea of Proof

Recall that

$$X_t^{\varepsilon} = X_0 + \int_0^t b(X_s^{\varepsilon}, Y_s^{\varepsilon})ds + L_t^1,$$

$$\bar{X}_t = X_0 + \int_0^t \bar{b}(\bar{X}_s)ds + L_t^1.$$

3. Idea of Proof

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$$X_t^\varepsilon = X_0 + \int_0^t b(X_s^\varepsilon, Y_s^\varepsilon)ds + L_t^1,$$
$$\bar{X}_t = X_0 + \int_0^t \bar{b}(\bar{X}_s)ds + L_t^1.$$

Thus

$$\begin{aligned} X_t^\varepsilon - \bar{X}_t = & \int_0^t \left[b(X_s^\varepsilon, Y_s^\varepsilon) - \bar{b}(\bar{X}_s)\right] ds \\ = & \int_0^t \left[b(X_s^\varepsilon, Y_s^\varepsilon) - \bar{b}(X_s^\varepsilon)\right] ds + \int_0^t \left[\bar{b}(X_s^\varepsilon) - \bar{b}(\bar{X}_s)\right] ds. \end{aligned}$$

Note that $\bar{b}$ is Lipschitz continuity, then for $p \in [1, \alpha)$,

$$\mathbb{E}\left(\sup_{t\in[0,T]}|X_t^{\varepsilon}-\bar{X}_t|^p\right) \leq C_p\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t b(X_s^{\varepsilon},Y_s^{\varepsilon})-\bar{b}(X_s^{\varepsilon})ds\right|^p\right] + C_{p,T}\mathbb{E}\int_0^T|X_t^{\varepsilon}-\bar{X}_t|^p dt.$$

By Gronwall's inequality, we get

$$\mathbb{E}\left(\sup_{t\in[0,T]}|X_t^{\varepsilon}-\bar{X}_t|^p\right) \leq C_{p,T}\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t b(X_s^{\varepsilon},Y_s^{\varepsilon})-\bar{b}(X_s^{\varepsilon})ds\right|^p\right].$$

Now, consider the following Poisson equation with parameter x:

$$-\mathscr{L}_2(x)\Phi(x,y) = b(x,y) - \bar{b}(x), \quad y \in \mathbb{R}^{d_2}, \tag{8}$$

where $\mathscr{L}_2(x)$ is the generator of the following frozen equation.

$$\begin{cases} dY_t^{x,y} = f(x, Y_t^{x,y})dt + dL_t^2, \\ Y_0^{x,y} = y. \end{cases} \tag{9}$$

Denote

$$\Phi(x,y) := \int_0^\infty \left[\mathbb{E}b(x, Y_t^{x,y}) - \bar{b}(x)\right] dt,$$

Then it is easy to prove that $\Phi(x,y)$ solves PDE (8).

Meanwhile, the solution $\Phi(x, y)$ satisfy the following estimates:

$$\sup_{x\in\mathbb{R}^{d_1}} |\Phi(x, y)| \leq C(1 + |y|), \quad \sup_{x\in\mathbb{R}^{d_1}, y\in\mathbb{R}^{d_2}} \|\nabla_y \Phi(x, y)\| \leq C, \tag{10}$$

$$\sup_{x\in\mathbb{R}^{d_1}} \|\nabla_x \Phi(x, y)\| \leq C_\theta(1 + |y|^{\theta}), \tag{11}$$

$$\|\nabla_x \Phi(x_1, y) - \nabla_x \Phi(x_2, y)\| \leq C|x_1 - x_2|^{\gamma}(1 + |x_1 - x_2|^{1-\gamma})(1 + |y|), \tag{12}$$

where $\theta \in (0, 1]$, $\gamma \in (\alpha - 1, 1)$.

By Itô's formula, we have

$$\Phi(X_t^\varepsilon, Y_t^\varepsilon) = \Phi(x, y) + \int_0^t \mathscr{L}_1(Y_r^\varepsilon)\Phi(X_r^\varepsilon, Y_r^\varepsilon)dr + \frac{1}{\varepsilon}\int_0^t \mathscr{L}_2(X_r^\varepsilon)\Phi(X_r^\varepsilon, Y_r^\varepsilon)dr + M_t^{\varepsilon,1} + M_t^{\varepsilon,2},$$

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where

$$\mathscr{L}_1(y)\Phi(x,y) := -(-\Delta_x)^{\alpha/2}\Phi(x,y) + \langle b(x,y), \nabla_x\Phi(x,y)\rangle;$$

$$M_t^{\varepsilon,1} := \int_0^t\int_{\mathbb{R}^{d_1}} \Phi(X_{r-}^\varepsilon + x, Y_{r-}^\varepsilon) - \Phi(X_{r-}^\varepsilon, Y_{r-}^\varepsilon)\tilde{N}^1(dr,dx);$$

$$M_t^{\varepsilon,2} := \int_0^t\int_{\mathbb{R}^{d_2}} \Phi(X_{r-}^\varepsilon, Y_{r-}^\varepsilon + \varepsilon^{-1/\alpha}y) - \Phi(X_{r-}^\varepsilon, Y_{r-}^\varepsilon)\tilde{N}^2(dr,dy).$$

As a result, it is easy to see

$$\int_0^t b(X_s^\varepsilon, Y_s^\varepsilon) - \bar{b}(X_s^\varepsilon)ds = \int_0^t -\mathscr{L}_2(X_r^\varepsilon)\Phi(X_r^\varepsilon, Y_r^\varepsilon)dr$$

$$= \varepsilon\left[\Phi(x, y) - \Phi(X_t^\varepsilon, Y_t^\varepsilon) + \int_0^t \mathscr{L}_1(Y_r^\varepsilon)\Phi(X_r^\varepsilon, Y_r^\varepsilon)dr + M_t^{\varepsilon,1} + M_t^{\varepsilon,2}\right].$$

As a result, it is easy to see

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Hence, we have

$$\mathbb{E}\left[\sup_{t\in[0,T]} |X_t^\varepsilon - \bar{X}_t|^p\right] \leq C\mathbb{E}\left[\sup_{t\in[0,T]} \left|\int_0^t \mathscr{L}_2(X_r^\varepsilon, Y_r^\varepsilon)\Phi(X_s^\varepsilon, Y_s^\varepsilon)ds\right|^p\right]$$

$$\leq C\varepsilon^p\left[\mathbb{E}\sup_{t\in[0,T]} |\Phi(x,y) - \Phi(X_t^\varepsilon, Y_t^\varepsilon)|^p + \mathbb{E}\int_0^T |\mathscr{L}_1(Y_r^\varepsilon)\Phi(X_r^\varepsilon, Y_r^\varepsilon)|^p\, dr\right.$$

$$\left. + \mathbb{E}\left(\sup_{t\in[0,T]} |M_t^{\varepsilon,1}|^p\right) + \mathbb{E}\left(\sup_{t\in[0,T]} |M_t^{\varepsilon,2}|^p\right)\right].$$

By estimates (10)-(12) and the following estimate:

$$\sup_{\varepsilon\in(0,1)} \mathbb{E}\int_0^T |\mathscr{L}_1(Y_r^\varepsilon)\Phi(X_r^\varepsilon, Y_r^\varepsilon)|^p\, dr \leq C_{p,T}(1+|x|^p+|y|^p);$$

$$\mathbb{E}\left(\sup_{t\in[0,T]} |\Phi(X_t^\varepsilon, Y_t^\varepsilon)|^p\right) \leq C_{p,T}(1+|y|^p)\varepsilon^{-\frac{p}{\alpha}};$$

$$\sup_{\varepsilon\in(0,1)} \mathbb{E}\left(\sup_{t\in[0,T]} |M_t^{\varepsilon,1}|^p\right) \leq C_p(1+|y|^p);$$

$$\mathbb{E}\left(\sup_{t\in[0,T]} |M_t^{\varepsilon,2}|^p\right) \leq C_{p,T}\varepsilon^{-\frac{p}{\alpha}}.$$

We final obtain

$$\mathbb{E}\left(\sup_{t\in[0,T]} |X_t^\varepsilon - \bar{X}_t|^p\right) \leq C_{p,T}(1+|x|^p+|y|^p)\varepsilon^{p(1-\frac{1}{\alpha})}.$$

Thank you very much!